

Adhesive Contact to a Coated Elastic Substrate

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Abstract. We show how the quasi-analytic method developed to solve linear elastic contacts to coated substrates (Perriot A. and Barthel E. *J. Mat. Res.*, **2004**, *19*, 600) may be extended to adhesive contacts. Substrate inhomogeneity lifts accidental degeneracies and highlights the general structure of the adhesive contact theory. We explicit the variation of the contact variables due to substrate inhomogeneity. The relation to other approaches based on Finite Element analysis is discussed.

1. Introduction

In elastic contact problems, it is known that the homogeneous substrate is a special case which leads to counterintuitive results. As an example, for *adhesionless contacts*, the relation between penetration δ and contact radius a is independent from the *mechanical parameters* (Young's modulus E , Poisson ratio ν , reduced modulus $E^* = E/(1 - \nu^2)$) and can be in all generality expressed as a function of the *shape* of the contacting bodies $z(r)$ only.

For instance, for a rigid cone of half included angle β in frictionless contact with an elastic substrate:

$$\delta = \frac{2}{\pi} \frac{a}{\tan \beta} \quad (1)$$

Obviously, the reduced modulus is absent and one might conclude that the relation has a geometrical origin. To show that this is not the case, one needs only remove the assumption of homogeneous substrate. Then, for a coated substrate, for instance, the penetration will again depend upon the mechanical parameters of the system, in a highly non trivial manner [1]. Overlooking the more complex penetration dependence upon contact radius might actually result in inaccuracies in the determination of thin film mechanical properties by nanoindentation [2, 3, 4, 5].

Similarly, for adhesive contacts, a counterintuitive result is that the *adhesion force* or pull-out force $F_{pullout}$ for a sphere of radius R and adhesion energy w is independent from the mechanical parameters. It is valued

$$F_{pullout} = n\pi w R \quad (2)$$

where various n values have been proposed such as 1 [6], 3/2 [7] or 2 [8].

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However, changing the geometry from sphere to cone [9] or removing the assumption of a homogeneous substrate, for example with a coated substrate [10, 11], again results in reintroducing the dependence upon mechanical parameters.

With the idea that the real structure of the elastic contact models only appears when considering a non homogeneous substrate, the present contribution aims at: 1) showing how the recently developed quasi analytical method for contacts to coated substrates [1] may be used to account for adhesion as previously used by Mary et al. [12] 2) exploring how the different concepts developed for homogeneous adhesive contacts may be applied to a *non homogeneous system*, to wit, a coated substrate.

Section 2 reviews the various concepts which have been developed to establish the relation between macroscopic contact variables F , δ and a and the physical phenomena of adhesion. Section 3 describes the surface stress transform method pioneered by Sneddon. Section 4 explores the application of the method to the adhesive contact to a homogeneous substrate, while section 5 explores the application of the method to coated substrates.

2. Adhesive elastic contact concepts and previous results on coated substrates

2.1. Elastic adhesive contact - problem description

Contact problems are characterized by mixed boundary conditions: the surface displacement is specified inside the contact zone and the normal stress at the surface outside. Note that this normal stress outside the contact zone is zero, except at the edge of the contact zone where more or less long range adhesive surface forces develop in the so-called interaction zone (Fig. 1).

Three concepts have been developed to handle these complex boundary conditions, provide a solution and establish relations between the macroscopic variables F , δ and a .

2.2. Three concepts to solve the adhesive elastic contact problem

2.2.1. Energy minimization The first method disregards the contact edge details, assumes a given form for the surface stresses inside the contact zone and calculates the solution parameters by energy minimization, taking into account the adhesion energy w as a πwa^2 term. This macroscopic approach was implemented as early as 1934, when Derjaguin assumed a Hertz-like stress distribution and combined it with an adhesion energy contribution proportional to the contact area, which resulted [6] in a pull-out force given by $n = 1$ in Eq. 2.

Based on elastomer adhesion experiments, a more realistic surface deformation built up from the addition of a flat punch displacement to the Hertz solution resulted in the JKR model [7] § where $n = 3/2$ in Eq. 2. This model, now generally accepted for soft

§ Note that an early version of this model can be found in the appendix of G. Sperling's PhD thesis [13],

solids, provides a more consistent description of the stresses inside the contact zone than the original 1934 result.

More generally, a formal method to minimize the total energy with regards to the contact radius a naturally leads to introducing the energy release rate, a concept borrowed from fracture mechanics : if \mathcal{E} is the stored elastic energy, the energy release rate G can be defined in an axisymmetric geometry as

$$2\pi a G(a) = \left. \frac{\partial \mathcal{E}}{\partial a} \right|_{\delta} \quad (3)$$

Such an approach was used for instance as an alternative derivation of the JKR model in [14].

2.2.2. Stress intensity factor The second method is to focus on the crack-like stress singularity at the contact edge. It is described in terms of the *stress intensity factor*. The relation between the energy release rate and the stress intensity factor was initially proposed by Irwin [15]. In the field of the adhesive contact, this second method was mainly propounded by Maugis [9, 16, 17]. It is particularly useful for direct extensions of the Linear Elastic Fracture Mechanics where a description in terms of stresses at the crack tip is necessary: for instance, it may be used to include crack tip viscoelasticity [16, 17, 18].

2.2.3. Exact solution The third method is actually more general: it takes into account the details of the attractive stress distribution outside the contact zone. This idea was somehow initiated by the DMT model [8] ($n = 2$ in Eq. 2) but was brought to a new dimension by the introduction of a cohesive zone model by Maugis [19], which resulted in a clear exposition of the transition between the two limiting models JKR and DMT. It also led to the development of the fully viscoelastic adhesive contact models [20, 21]. In these models, stress relaxation inside the contact zone – as in the adhesionless viscoelastic contact by Ting [22] – and creep inside the interaction zone – as in the viscoelastic crack models [16, 17, 18] – are taken into account simultaneously.

Just as the viscoelastic case extends the adhesive contact model from instantaneous response to time dependent constitutive relation, the coated substrate generalizes the model to an inhomogeneous half space. Our interest here will however be limited to the small cohesive zone sizes, located on the JKR side of the transition, which connect this approach to the previous two. For an extension to the JKR-DMT transition with a coated substrate, when the interaction zone grows to extensions comparable or larger than the film thickness, in the spirit of the Maugis model, see [23].

where it stands as a curiosity, not as the "Theorie der Haftung" developped in the body of the text, which mainly deals with surface roughness and plastic deformation.

2.3. Review of previous results on the adhesive contact to coated substrates

On the theoretical side, the literature on the adhesive contact to coated substrates is scanty, although, in practice, thin soft adhesive layers are often used, as in many applications of pressure sensitive adhesives and functional or protective organic coatings. Actually only two series of papers by Shull and coworkers [24, 25] on one side and Johnson and Sridhar [10, 11] on the other side have been published.

Shull and coworkers calculate the energy release rate from the contact compliance (cf. 5.2.2). To complete the calculations, the compliance of coated substrates are computed by Finite Element (FE) methods and phenomenological fits or numerical differentiation may be carried out.

Johnson and Sridhar's approach goes along the line of the stress intensity factor (section 2.2.2) approach: still using FE, they directly calculate the energy release rate G by a stiffness matrix variation method [26] and extract stress intensity factors from G . From this stress intensity factor, they build the adhesive contact relations.

Both methods converge on their ultimate use of Finite Element calculations to handle the complexities of the coated substrate response. Appearances to the contrary, these two series of papers seem actually to rely on the same method and an in depth investigation of the Parks method in the context of the frictionless contact would probably connect the stiffness matrix derivative used by Parks to the contact stiffness derivative method as developed by Shull.

In the present paper, we show that the quasi-analytical method introduced previously [1] may be used to efficiently compute all the necessary contact variables and in particular provide direct numerical evaluations for the stress intensity factor for coated substrates. It is also useful to expose the equivalence of the various concepts for the adhesive contact in the wider context of non-homogeneous substrates.

3. Sneddon and the surface stress transform

Adequate transforms facilitate the investigation of linear elastic axisymmetric contacts formulated as integral equations [27]. It is ideally suited to the adhesive contacts because of the relevant boundary conditions including the interaction zone.

Following our previous studies [20, 21, 28, 29], we use $g(r)$, which is a transform of the normal stress distribution at the surface σ_z defined as

$$g(r) = \int_r^{+\infty} \frac{s\sigma_z(s)}{\sqrt{s^2 - r^2}} ds \quad (4)$$

and $\theta(r)$ which is a transform of the normal surface displacement u_z defined by

$$\theta(r) = \frac{\partial}{\partial r} \int_0^r \frac{su_z(s)}{\sqrt{r^2 - s^2}} ds \quad (5)$$

These relations are suited to the adhesive contact problem because $g(r)$ is expressed as a function of normal surface stresses at radii values larger than r and $\theta(r)$ as a function of surface normal displacement at radii values smaller than r , in agreement

with the adhesive contact boundary conditions (Fig. 1). In particular g is zero outside the interaction zone.

Similarly, inside the contact zone ($r \leq a$), $u_z(r) = \delta - z(r)$ where $z(r)$ is the shape of the indenter. Integration by parts transforms Eq. 5 into

$$\theta(r) = \delta - \delta_0(r). \quad (6)$$

where δ is the penetration and δ_0 depends only upon the shape of the indenter $z(r)$ through

$$\delta_0(r) = r \int_0^r \frac{z'(s)ds}{\sqrt{r^2 - s^2}} \quad (7)$$

The full power of these transforms appears when it is recognized [20, 21] that in the linear elastic case and for a *homogeneous* substrate, mechanical equilibrium results in

$$g(r) = \frac{E^*}{2} \theta(r) \quad (8)$$

This provides the direct solution to the adhesive contact problem which we detail now.

4. Adhesive contacts on homogeneous substrates

Before tackling the problem of the adhesion to coated substrates we review our present understanding of elastic contacts to homogeneous substrates using the transform method.

4.1. Non adhesive contact of smooth indenters

In the absence of adhesion, Eq. 4 shows that $g(a) = 0$. If the indenter shape $z(r)$ is smooth at the contact edge, then Eqs. 6 and 8 imply $\theta(a) = 0$ and

$$\delta = \delta_0(a) \quad (9)$$

Thus the indenter penetration in an adhesionless contact is the function $\delta_0(a)$, which depends only upon the geometry of the indenter (Eq. 7) and not on the elastic parameters. This is the counterintuitive result stated in section 1.

4.2. Non adhesive contact of flat punches

A special case however is the flat punch for which $\delta_0(r) = 0$ for $r < a$ and $g(r) = 0$ for $r > a$. The penetration is independent of the contact radius as expected since the contact radius is constant, equal to the punch radius.

Then $\theta(a^-) = \delta$ so that $g(a^+) = 0$ and $g(a^-) \neq 0$. The exact meaning of this discontinuity in the g transform at a will be discussed below (section 4.4.2).

Note also that the force can be obtained through the simple but general expression [29]

$$F = 4 \int_0^{+\infty} g(s)ds \quad (10)$$

Specifically here,

$$F_0(a) = 4 \int_0^a g(s) ds \quad (11)$$

For the flat punch the force is then directly calculated from Eqs. 6, 8 and 10 as

$$F_0(s) = S(a)\delta_{fp} \quad (12)$$

with

$$S(a) = 2aE^*. \quad (13)$$

4.3. Adhesive contacts

In the adhesive case, however, although the indenter shape is smooth at the contact edges, the condition $g(a) = 0$ is relaxed due to the adhesive interactions. The core of the JKR approximation is to neglect the details of the stresses and deformations at the contact edge and encapsulate the adhesive contribution in an additional flat punch displacement. This amounts to a description where the attractive interaction stresses drop abruptly to zero outside the contact zone, leading to a stress singularity akin to those met in fracture mechanics.

Then Eqs. 9 and 11 become

$$\delta = \delta_0 + \delta_{fp} \quad (14)$$

$$F = F_0 + S(a)\delta_{fp} \quad (15)$$

The question is to provide a relation between this additional flat punch displacement δ_{fp} and the adhesion energy w for a given contact radius a .

4.4. Various concepts for the derivation of adhesive contact results

For future reference, we now apply the surface stress transform solution to the three methods developed to handle adhesive contacts (section 2.2).

4.4.1. Energy Release Rate A very compact derivation is obtained when it is recognized that for a homogeneous substrate g et θ are independent of a inside the contact zone. In addition the total mechanical energy \mathcal{E} as a function of the normal surface stress distribution is [29]

$$\mathcal{E} = \frac{4}{E^*} \int_0^{+\infty} ds g^2(s) = E^* \int_0^{+\infty} ds \theta^2(s) \quad (16)$$

Then for constant penetration, with $g(a) = 0$ outside the contact zone,

$$2\pi a G(a) = \frac{4g(a)^2}{E^*} = E^* \theta(a)^2 \quad (17)$$

Equating G and the adhesion energy w results in

$$w = \frac{2g(a)^2}{\pi E^* a} \quad (18)$$

The energy release rate $G(a)$ can be expressed as a function of the local stress distribution at the crack tip because of Eq. 4 and $g(r) = 0$ for $r > a$.

4.4.2. Stress intensity factor This expression together with the plain strain Irwin relation [9, 15] between stress intensity factor and energy release rate||

$$G = K_I^2 / (2E^*) \quad (19)$$

suggests that $2g(a)/\sqrt{\pi a}$ assumes the role of a stress intensity factor.

A direct derivation is obtained [9] if we use the following expression for the stress intensity factor at the contact edge

$$K_I = \lim_{\epsilon \rightarrow 0} (2\pi\epsilon)^{1/2} \sigma(a - \epsilon) \quad (20)$$

Eq. 4 can be inverted [29] providing

$$\sigma_z(s) = \frac{2}{\pi} \left[\frac{g(a)\Theta(a-s)}{(a^2-s^2)^{1/2}} + \int_s^{+\infty} dt \frac{g'(t)}{(t^2-s^2)^{1/2}} \right] \quad (21)$$

where it has been assumed that g is smooth except for a discontinuity at a . We use the notation $g(a^-) = g(a)$ and assume $g(a^+) = 0$. Then

$$K_I = \frac{2g(a)}{\sqrt{\pi a}} \quad (22)$$

which confirms that $g(a)$ has the form and meaning of a stress intensity factor.

As a stress intensity factor, $g(a)$ is a measure of stress singularity at the contact edge. An ancillary property of the transform defined by Eq. 4 is that it regularizes the singular crack like stress distributions and Eq. 5 establishes a relation between the additional flat punch penetration and the stress singularity through

$$\theta(a) = \delta_{fp} = \frac{2g(a)}{E^*} \quad (23)$$

4.4.3. Self-consistent method The self-consistent description of the interaction zone [19, 28, 30] can be explored at the limit of negligible interaction zone extension, *i.e.* in the JKR limit.

One possible starting point of the self-consistent method [28, 29] is to calculate the adhesion energy by

$$w = - \int_a^{+\infty} \sigma(s) \frac{dh}{ds} ds \quad (24)$$

Following [21](Eq 15) the gap between the surfaces h is split into the contributions of the contact stresses and the interaction stresses

$$h(r) = h_{Hertz}(r, a) + h_{int}(r, a) \quad (25)$$

If the interaction range is small (which results in $c-a \ll a$), the radial extension of the interaction will also be small, g' is peaked around a , and the dominant term in Eq. 24 will be the $\partial u_{z,int}/\partial s$ term. Thus [29],

$$w \simeq - \int_a^c \sigma_s \frac{dh_{int}}{ds} ds \simeq - \frac{4}{\pi a E^*} \int_a^{+\infty} g'(t) g(t) dt \quad (26)$$

As a result, one again recovers Eq. 18. Combined with Eq. 22, this method may be viewed as a direct derivation of the Irwin relation [15].

|| The 1/2 factor comes from the rigidity of the punch.

4.5. Normalized form

The adhesive contact equations are obtained by combining Eqs. 14-15 and Eq. 18. For later reference, we specialize the results to the spherical indenter and use the Maugis normalization [19]

$$P = \frac{F}{\pi w R} \quad (27)$$

$$D = \frac{\delta}{\left(\frac{\pi^2 w^2 R}{E^*}\right)^{1/3}} \quad (28)$$

$$\bar{a} = \frac{a}{\left(\frac{\pi w R^2}{E^*}\right)^{1/3}} \quad (29)$$

One obtains

$$P = \frac{4}{3}\bar{a}^3 - 2\sqrt{2}\bar{a}^{3/2} \quad (30)$$

$$D = \bar{a}^2 - \sqrt{2}\bar{a}^{1/2} \quad (31)$$

These are the usual JKR equations which will be generalized in the next section. Their structure directly reflects the general structure of Eqs. 14-15 with the identifications $\bar{\delta}_{fp} = -\sqrt{2}\bar{a}^{1/2}$ and $\bar{S}_{fp} = 2\bar{a}$.

5. Coated Substrates – Thin Films

5.1. Description of the contact

If the substrate is not homogeneous the simple diagonal equilibrium relation Eq. 8 between g and θ is lost. However, keeping the same transforms, we have shown [1] that a useful relation subsists

$$\theta(r; a, t, [E]) = \frac{2g(r; a, t, [E])}{E_1^*} + \frac{2}{\pi} \int_0^{+\infty} g(s; a, t, [E]) K(r, s; t, [E]) ds \quad (32)$$

where K expresses the elastic response of the coated substrate [1], t stands for the coating thickness and $[E]$ for the four mechanical parameters (film and substrate Young's moduli and Poisson ratios). Since g is zero outside the contact zone, the upper boundary will typically be the contact radius a or the interaction zone radius c if $c \neq a$. The finite integration interval facilitates the numerical inversion [1] of Eq. 32. Note that direct analytical inversion is impossible because of the complex expression for K .

We now split the stress distribution and the displacement field into their non adhesive δ_H and flat punch δ_{fp} components.

For the non adhesive contact of an indenter of regular shape, the penetration is given by

$$\delta_H(a, t, [E]) - \delta_0(r) = \frac{2g_H(r; a, t, [E])}{E_1^*} \quad (33)$$

$$+ \frac{2}{\pi} \int_0^a g_H(s; a, t, [E]) K(r, s; t, [E]) ds \quad (34)$$

where $\delta_0(r)$ reflects the indenter shape and $g_H(a; a, t, [E]) = 0$ in the absence of adhesion, ensuring a unique solution. Eq. 34 generalizes Eq. 15 and introduces the non-trivial relation mentioned in section 1.

For the contact of the *flat punch* the problem is linear with δ_{fp} and

$$\delta_{fp} = \frac{2g_{fp}(r; a, t, [E])}{E_1^*} + \frac{2}{\pi} \int_0^a g_{fp}(s; a, t, [E]) K(r, s; t, [E]) ds \quad (35)$$

with $g_{fp}(r; a, t, [E]) \propto \delta_{fp}$ and $g(a; a, t, [E]) \neq 0$. Eq. 35 generalizes Eq. 23 into a non trivial proportionality relation between $g(a)$ and the flat punch penetration δ_{fp} .

Then the penetration for the adhesive contact of an indenter of arbitrary shape is

$$\delta(a, t, [E]) = \delta_H(a, t, [E]) + \delta_{fp} \quad (36)$$

With the following definition of the contact stiffness

$$S(a, t, [E]) = \frac{F_{fp}(a, t, [E])}{\delta_{fp}} \quad (37)$$

the force is

$$F(a, t, [E]) = F_H(a, t, [E]) + S(a, t, [E]) \delta_{fp} \quad (38)$$

Equations 36 and 38 generalize Eqs. 15 and 14. Here again, the question is to provide a relation between the additional flat punch displacement δ_{fp} and the adhesive interaction.

5.2. Derivation

For coated substrates, the resolution benefits from more general expressions derived for viscoelastic adhesive contacts [31]. This is not fortuitous but results from a similar breakdown of the simplified relations derived through Eq. 8 when spatial heterogeneity or time dependence is introduced.

5.2.1. Energy Release Rate The total energy is [31]

$$\mathcal{E} = 2 \int_0^{+\infty} ds g(s) \theta(s) \quad (39)$$

Then

$$2\pi a G(a) = 2 \left(g(a) \theta(a) + \int_0^a ds \frac{dg(s)}{da} \theta(s) \right) \quad (40)$$

because $G(a)$ is calculated at constant displacement so that inside the contact zone, the surface displacement – and therefore $\theta(r)$ – are unaffected by the additional stress distribution $g(a)$: there is only a stress rearrangement inside the contact zone which does affect $g(r)$, $r < a$ for a coated substrate: indeed the local relation Eq. 8 between g and θ breaks down in this case.

Following Mary et al. [12], one can show that this non local contribution is actually canceled by the non local contribution included in the $g(a)\theta(a)$ term. For $s < a$

$$-\frac{\pi}{E_1^*} \frac{dg(s)}{da} = g(a) K(a, s) + \int_0^a dr \frac{dg(r)}{da} K(r, s) \quad (41)$$

and multiplying by $g(s)$ and integrating between 0 and a , one obtains

$$g(a)\theta(a) + \int_0^a dr \frac{dg(r)}{da} \theta(r) = \frac{2g(a)^2}{E_1^*} \quad (42)$$

The region affected by the crack tip stresses extend over a distance commensurate with the interaction zone size, *i.e.* it is smaller than the film thickness so that the energy release rate is controlled by the *film* compliance.

5.2.2. Compliance Method The compliance formulation is at the core of the method used by Shull and coworkers but the derivation given in some of their earlier papers was obscured by unnecessary assumptions. This formulation emerges readily when \mathcal{E} in Eq. 39 is expressed as a function of strain instead of stress¶.

We split the total stress and strain fields in their non adhesive (g_H and θ_H) and flat punch components (g_{fp} and θ_{fp}). This allows for easy integration because $\theta_{fp}(r; a, t, [E]) = \delta_{fp}$ is independent of r . Using Betti's theorem to calculate the cross terms

$$2 \int_0^{+\infty} ds g_H(s; a, t, [E]) \theta_{fp}(s; a, t, [E]) \quad (43)$$

$$= 2 \int_0^{+\infty} ds g_{fp}(s; a, t, [E]) \theta_H(s; a, t, [E]) \quad (44)$$

$$= \frac{1}{2} \delta_{fp} P_H(a, t, [E]) \quad (45)$$

and similarly using Eq. 37

$$2 \int_0^{+\infty} ds g_{fp}(s; a, t, [E]) \theta_{fp}(s; a, t, [E]) = \frac{1}{2} S(a, t, [E]) \delta_{fp}^2 \quad (46)$$

to express the flat punch elastic energy, one obtains the total energy

$$\mathcal{E}(a, \delta_{fp}) = U_H(a, t, [E]) + \frac{1}{2} S(a, t, [E]) \delta_{fp}^2 + P_H(a, t, [E]) \delta_{fp} \quad (47)$$

where δ_{fp} is negative.

A graphic illustration of this result is given in Fig. 2.

The energy release rate G is the differential of the total energy $\mathcal{E}(a, \delta_{fp})$ with respect to contact area (Eq. 3) at constant total penetration δ where

$$\delta = \delta_H(a) + \delta_{fp} \quad (48)$$

$$\left. \frac{d\mathcal{E}}{da} \right|_{\delta} = \frac{\partial \mathcal{E}}{\partial a} + \frac{\partial \mathcal{E}}{\partial \delta_{fp}} \frac{d\delta_{fp}}{da} \quad (49)$$

Now

$$dU_H = P_H(a, t, [E]) d\delta_H \quad (50)$$

$$dP_H = S(a, t, [E]) d\delta_H \quad (51)$$

¶ Such a *strain* energy release rate has also been introduced previously in viscoelastic crack problems by Schapery [32] and Greenwood and Johnson [18].

so that finally all terms cancel⁺ except the differential of the flat punch elastic energy and

$$2\pi a G(a) = \frac{1}{2} \delta_{fp}^2 \frac{dS}{da} \quad (52)$$

This differentiation process is also illustrated on Fig. 2.

5.2.3. Stress Intensity Factor The derivation of the stress intensity factor expression in section 4.4.2 is indendent of the material properties and only results from the definition of the transform g . It is therefore unchanged in the more general case and the pending problem is actually the relation between $g(a)$ and the energy release rate G .

5.2.4. Self-consistency In Eq. 25,

$$h_{int}(r, a, t, [E]) = \frac{2}{\pi} \int_a^r \frac{\theta(s) - \theta(a)}{\sqrt{r^2 - s^2}} ds \quad (53)$$

If at the crack tip the deformation predominantly results from the adhesive interactions – and this is the essence of the JKR limit – then Eq. 24 becomes [20]

$$w = -\frac{2}{\pi a} \int_a^c \theta'(r) g(r) dr \quad (54)$$

From Eq. 32 we obtain

$$\theta'(r) = \frac{2g'(r)}{E_1^*} + \frac{2}{\pi} \int_0^{+\infty} g(s) \frac{dK(r, s)}{dr} \quad (55)$$

The second term is well behaved when $c \rightarrow a$ since K is the elastic response of the coated substrate: the contact edge singularity results from the boundary conditions, not the response function. Thus it is g' which is peaked around a [28] so that in the end for $c < r < a$

$$\theta'(r) \simeq \frac{2g'(r)}{E_1^*} \quad (56)$$

and *again*

$$w = \frac{2g(a)^2}{\pi E^* a} \quad (57)$$

even for a non-homogeneous substrate.

The local response is indeed dominated by the local stress distribution and the *film* compliance: the same relation Eq. 17 between adhesion energy and stress intensity factor applies.

⁺ Due to the counterintuitive results obtained for the homogeneous substrate, one may develop doubts about the identity of the stiffness defined by Eq. 51 and the stiffness defined by Eq. 37. That this identity does hold for a non homogeneous substrate is shown in the appendix (section 8. It is due to the absence of adhesion: the normal stress at the edge of the contact is zero so that contact radius variation does not result in force variation, to first order.

5.3. Relation between $g(a)$ and S

Comparing Eqs. 52 and 57, one infers

$$\frac{4g(a)^2}{E_1^*} = \frac{1}{2} \delta_{fp}^2 \frac{dS}{da} \quad (58)$$

This is a remarkable result which one should in principle be able to derive from Eq. 35.

Note however that the present quasi-analytical method [1] simultaneously provides the flat punch stiffness and contact edge stress singularity $g(a)$ (or the stiffness derivative through Eq. 35) by the simple resolution of a linear system.

5.4. Normalized Solution

The procedure to compute a full force curve for the adhesive contact to coated substrates is therefore: for a given value of the contact radius a

- (i) compute the adhesionless penetration and force for the given indenter shape
- (ii) compute the force and stress intensity factor (or $g(a)$) for the flat punch for a unit value of the penetration
- (iii) compute actual value of $g(a)$ from Eq. 57 and rescale flat punch force and penetration
- (iv) compute the solution from Eqs. 36 and 38

This provides the solution under the form of two relations between the local deformation (stress intensity factor) and the local response on the one hand and the remote loading and the macroscopic response (contact stiffness) on the other hand.

In normalised form, with

$$\tilde{r} = \frac{r}{t} \quad (59)$$

one has*

$$F_{0,s} = \frac{a^3 E_1^*}{2R} \Pi_s(\tilde{a}, t, [E]) \quad (60)$$

$$\delta_{0,s} = \frac{a^2}{R} \Delta_s(\tilde{a}, t, [E]) \quad (61)$$

$$S = 2a E_1^* \mathcal{E}_{eq}(\tilde{a}, t, [E]) \quad (62)$$

$$g(r; a, t, [E]) = \frac{\delta_{fp} E_1^*}{2} \Gamma(\tilde{r}; \tilde{a}, t, [E]) \quad (63)$$

where the normalized variables can be numerically calculated by the simple algorithm presented in [1].

* For a cone:

$$F_{0,c} = \frac{\pi a^2 E_1^*}{4 \tan \beta} \Pi_c(\tilde{a}, t, [E])$$

$$\delta_{0,c} = \frac{\pi a}{2 \tan \beta} \Delta_c(\tilde{a}, t, [E])$$

All variables equal 1 for the homogeneous substrate except $\Pi_s = 8/3$ for the sphere \ddagger .

$\Gamma(\tilde{r}; \tilde{a}, t, [E])$ is the surface stress transform normalized to penetration δ . In particular, $\Gamma(\tilde{a}; \tilde{a}, t, [E])$, denoted $\Gamma(1)$ below for brevity, is the contact edge singularity $g(a; a, t, [E])$ incurred for a coated system – normalized to a homogeneous material with the film elastic properties – at identical δ_{fp} value. The variable $\Gamma(1)$, which is positive since both δ_{fp} and $g(a)$ are negative in Eq. 63, is a function of \tilde{a} , and depends upon the mechanical parameters of the system. From Eq. 52 the following identity holds:

$$\Gamma(1) = \sqrt{\frac{1}{2E_1^*} \frac{dS}{da}} = \sqrt{\frac{d(a\mathcal{E}_{eq})}{da}} \quad (64)$$

Then for the sphere, keeping the Maugis normalization, one introduces the film thickness normalized to the typical adhesive contact radius

$$\eta = \frac{t}{\left(\frac{\pi w R^2}{E_1^*}\right)^{1/3}} \quad (65)$$

and with $\bar{a} = \eta\tilde{a}$ one obtains

$$\Pi_s = (\eta\tilde{a})^3 \frac{\Pi_{s,0}}{2} - 2\sqrt{2}(\eta\tilde{a})^{3/2} \frac{\mathcal{E}_{eq}}{\Gamma(1)} \quad (66)$$

$$D_s = (\eta\tilde{a})^2 \Delta_{s,0} - \sqrt{2}(\eta\tilde{a})^{1/2} \frac{1}{\Gamma(1)} \quad (67)$$

These equations generalize Eqs. 30-31. The homogeneous substrate force and penetration terms are corrected by the coated system factors $\Pi_{s,0}$ and $\Delta_{s,0}$, the homogeneous contact stiffness by \mathcal{E}_{eq} and, for identical stress intensity factor, the homogeneous penetration is corrected by $1/\Gamma(1)$.

6. Examples of Numerical Results

6.1. Reduced variables

For given mechanical parameters E_1^*/E_0^* , ν_0 and ν_1 one may calculate the four variables $\Pi_{s,0}, \Delta_{s,0}, \mathcal{E}_{eq}$ and $\Gamma(1)$ as a function of $\tilde{a} = a/t$. Typical results are illustrated in Fig. 3 for $E_1^*/E_0^* = 10$ and Fig. 4 for $E_1^*/E_0^* = 0.1$. The results for these reduced variables compare well with the FE calculations by Sridhar and Johnson [11].

The thin film contact problem exhibits a transition between film dominated to substrate dominated behaviour. In each limit cases, the contact behaves like a contact to a homogeneous system. The non trivial behaviour is apparent in the transition which, roughly speaking, occurs for $\tilde{a} \simeq 1$, but is shifted to higher values for compliant films (Fig. 3) and to lower values for stiff films (Fig. 4) [1].

\mathcal{E}_{eq} exhibits a transition between film to substrate reduced modulus as the contact radius increases is consistent with numerous previous works [33]. The behaviour of $\Delta_{s,0}$, which deviates from 1 in the midst of the transition [1, 5] has been much less studied, as

\ddagger For a cone $\Pi_c = 2$

mentioned previously (section 1). Similarly for the flat punch we note that $\Gamma(1)$ tracks \mathcal{E}_{eq} at small contact radius values, up to about the middle of the transition. This does result from Eq. 64 as a linear expansion of \mathcal{E}_{eq} shows (note that $\mathcal{E}_{\text{eq}}(0) = 1$). For larger contact radius values, higher order terms come into play and $\Gamma(1)$ starts to deviate from \mathcal{E}_{eq} .

6.2. Adhesive contact solutions

The reduced variables $\Pi_s(\tilde{a}, t, [E])$, $\Delta_s(\tilde{a}, t, [E])$, $\mathcal{E}_{\text{eq}}(\tilde{a}, t, [E])$ and $\Gamma(1) = \Gamma(\tilde{a}; \tilde{a}, t, [E])$ provide the normalized solution (Eqs. 66 and 67) as a function of η for a set of mechanical parameters. It is worth emphasizing that – provided the reduced variables have been calculated on a wide enough range of values of $\tilde{a} = a/t$ – the solution for any values of contact radius, film thickness and adhesion energy can be calculated from the reduced variables through the elementary arithmetics of the normalization Eqs. 66 and 67. This should provide for an easy algorithm to fit experimental data for adhesive contact on thin films.

Results for various values of η are displayed on Fig. 5 and 6 for $E_1^*/E_0^* = 10$ and Fig. 7 for $E_1^*/E_0^* = 0.1$ with $\nu_0 = \nu_1 = 0.25$.

For a coated substrate in the presence of adhesion, the system undergoes two transitions. The first one is the transition between coating and substrate response [1], when $\tilde{a} \simeq 1$ as described in the previous section. The second one is the transition from adhesion dominated to non adhesive contact as the load and therefore the contact radius increase [34]. This transition occurs when $\tilde{a} \simeq 1$. The deviation from the JKR results occur when the two transitions are simultaneous, *i.e.* when $\eta \simeq 1$.

Explicitly, for large values of η (*i.e.* thick films), the adhesive stage of the contact will occur for small \tilde{a} and the contact is film dominated and behaves as a homogeneous material with the mechanical parameters of the film.

For small values of η (*i.e.* thick films), the adhesive stage occurs for large \tilde{a} and the contact is substrate dominated: the contact behaves as a homogeneous material with the mechanical parameters of the substrate. However, at the edge of the contact, this contact is not equivalent to an adhesive contact to a homogeneous material with substrate properties. Indeed, the stress intensity factor is dominated by the film properties (Eq. 57).

As a result, moderate deviations on the pull-out force are evidenced. For soft layers, the adhesion force is enhanced [11]. For rigid layers there is a small reduction of the adhesion force^{††}.

^{††}This is not, however, the reason why on soft materials a rigid layer may drastically reduce the adhesion, which is often due to the suppression of non elastic additional dissipation phenomena. Note that here the adhesive force reduction is accompanied by an increase of the stress intensity factor.

7. Conclusion

The surface stress transform is adequate to handle the adhesive contact to coated substrates. It provides a numerically simple method to compute the four quantities necessary for the actual description of such adhesive contacts. For a given layer and substrate mechanical properties, fits to data with free adhesion energies and coating thicknesses may be performed easily. It also allows a consistent description of the crack tip which connects the various concepts developed in adhesive contact problems.

The method also allows the inclusion of finite indenter stiffness, a question which arises in practice and which will be studied in more details in a subsequent paper.

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8. Appendix: contact stiffness and flat punch stiffness

The identity of the contact stiffnesses defined by Eq. 37 and by Eq. 51 may look questionable: is the contact stiffness for an adhesionless curved indenter identical with the flat punch stiffness for an identical contact radius, even for an inhomogeneous substrate ?

Since the curved indenter is adhesionless, $g(a) = 0$. Then from Eq. 10 we have

$$\frac{dP}{d\delta} = 4 \int_0^a \frac{dg(r)}{d\delta} dr \quad (68)$$

From the differentiation of Eq. 32 with δ together with Eq. 5,

$$1 = \frac{2}{\pi} \int_0^a \frac{dg(r)}{d\delta} K(r, s; t, [E]) dr \quad (69)$$

This is the flat punch equilibrium equation for unit penetration. Then the force

$$F_{fp} = 4 \int_0^a \frac{dg(r)}{d\delta} dr \quad (70)$$

is the flat punch force for unit penetration, *i.e.* the flat punch stiffness $S(a)$. Therefore for an adhesionless curved indenter

$$\frac{dP}{d\delta} = S(a) \quad (71)$$

Note however that in general this stiffness is *not* the hertzian $3PR/4a^3$.

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Captions

Fig. 1: schematics of an adhesive contact: the contact zone (radius a) is surrounded by the interaction zone. The gap between surfaces outside of the contact zone is $h(r)$. The normal stress distribution turns from compressive at the center of the contact to tensile on both sides of the contact edge. The transform $\theta(r_1)$ is calculated from the normal surface displacement at $r < r_1$. The transform $g(r_2)$ is calculated from the normal surface stress for $r > r_2$.

Fig. 2: Adhesive contact configuration obtained from the adhesionless contact by a flat punch displacement δ_{fp} along the tangent to the adhesionless force F -penetration δ curve. Integration gives the stored elastic energy. The energy release rate is obtained as the variation with contact radius a of the stored elastic energy at constant total penetration δ (shaded area).

Fig. 3: Normalized force Π_s (left) and penetration Δ_s (right) for an adhesionless sphere as a function of contact radius normalized to film thickness $\tilde{a} = a/t$. The film to substrate modulus ratio is 0.1 and Poisson ratio 0.25 for both materials. The contact stiffness E_{eq} (left) and contact edge stress intensity factor $\Gamma(1)$ (left) are calculated from the flat punch boundary conditions. For $\tilde{a} \simeq 1$ the transition from film dominated to substrate dominated contact occurs.

Fig. 4: Same plot as Fig. 3 for a film to substrate modulus ratio equal to 10. The transition occurs earlier due to film rigidity.

Fig. 5: Normalized force versus normalized contact radius as a function of η , the ratio of the film thickness to the typical zero load adhesive contact radius. When $\eta \simeq 1$, adhesive effects are significant in the transition from film to substrate dominated regimes: deviations from the JKR model are observed and increased adhesion forces are observed.

Fig. 6: Normalized force versus normalized penetration as a function of η . Identical parameters as in Fig. 5.

Fig. 7: Similar plot as in Fig. 5 with a film to substrate modulus ratio of 10. In the intermediate regime $\eta \simeq 1$, adhesion forces are reduced.

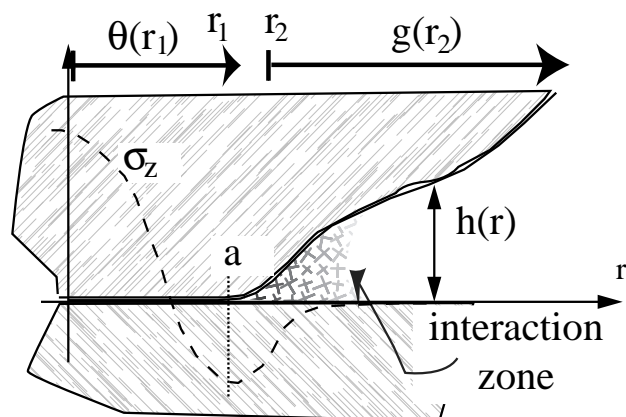


Figure 1.

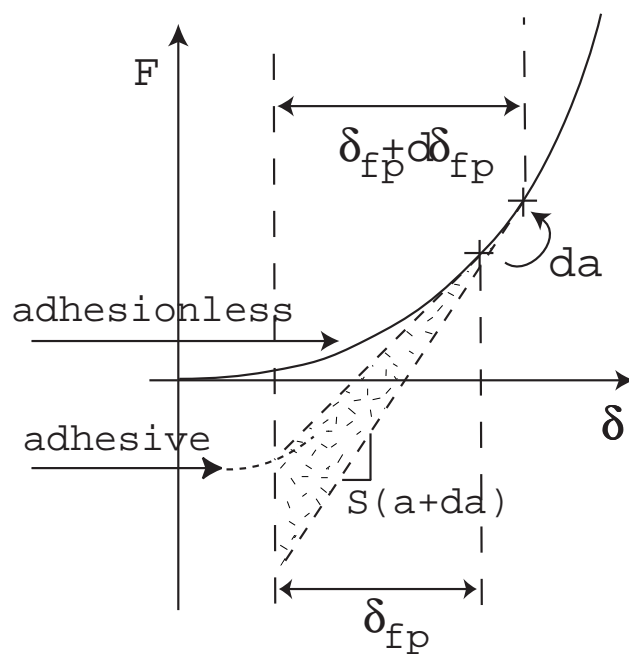


Figure 2.

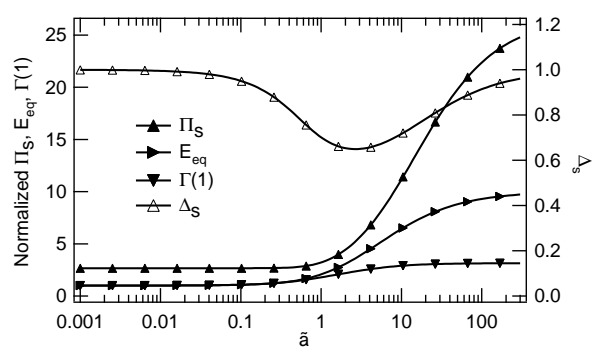


Figure 3.

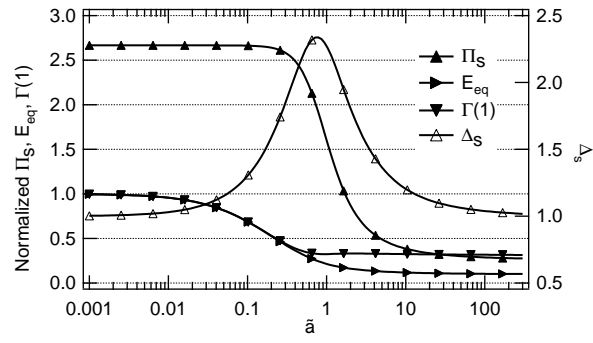


Figure 4.

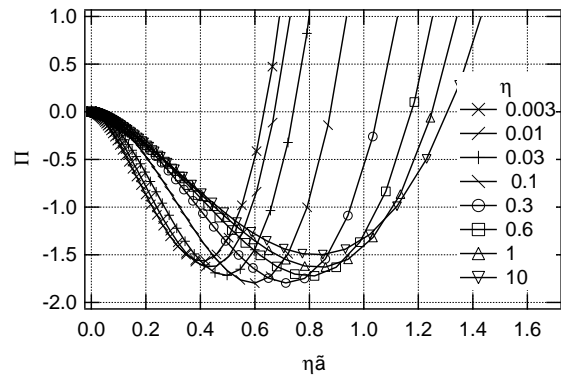


Figure 5.

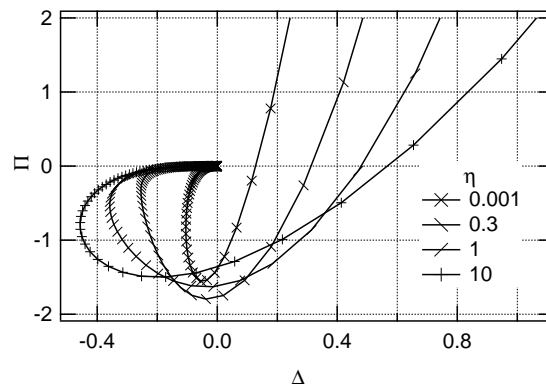


Figure 6.

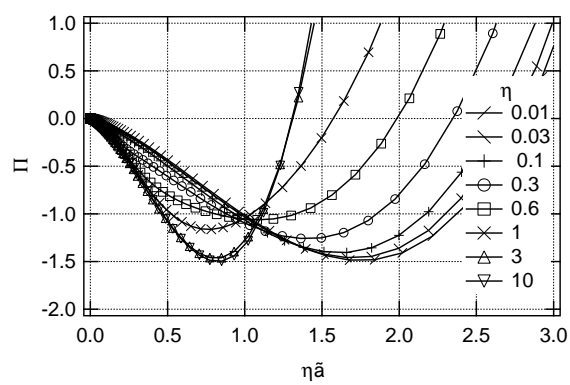


Figure 7.